Sequences and Limits

A sequence (x_n) of real numbers is said to be

i) bounded above if $\exists\, u \in \mathbb{R}$ s.t. $x_n \leq u \ \forall\, n \in \mathbb{N}$;

ii) bounded below if $\exists\, l \in \mathbb{R}$ s.t. $l \leq x_n \ \forall\, n \in \mathbb{N}$;

iii) bounded if both i) & ii), equivalent if $\exists\, \alpha \in \mathbb{R}$ s.t. $|x_n| \leq \alpha \ \forall\, n \in \mathbb{N}$ (where

$$|x| = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -x & \text{if } x < 0 \end{cases}$$

(iv) increasing if $x_n \leq x_{n+1} \ \forall\, n \in \mathbb{N}$

(v) decresing if $x_n \geq x_{n+1} \ \forall\, n \in \mathbb{N}$

$\Big\}$ monotone

(vi) convergent if $\exists\, l \in \mathbb{R}$ s.t (x_n) converges to l: $\forall\, \varepsilon > 0, \exists\, N \in \mathbb{N}$ s.t. $|x_n - l| < \varepsilon \ \forall\, n \geq N$.

(when such l exists, it is unique).

(vii) Cauchy if $\forall\, \varepsilon > 0 \ \exists\, N \in \mathbb{N}$ s.t. $|x_n - x_m| < \varepsilon \ \forall\, m, n \geq N$.

Exercises.

1. State the negation for each of the above.
2. Show the uniqueness remark in (vi). Hint: make use of the "Make life Easier Lemma": If $r \in \mathbb{R}$ is s.t. $|r| \leq \varepsilon \ \forall\, \varepsilon > 0$ then $r = 0$ (should $r \neq 0$ one would have $\varepsilon_0 := |r|/2 > 0$ but $|r| \nleq \varepsilon_0$.

3. (vi) $\Rightarrow$ (vii) ("$\Leftarrow$" also true but the proof is not easy at this stage)

4. (iv) $\Rightarrow$ (ii) & (v) $\Rightarrow$ (i).

5. Let (x_n) be an increasing seq and bounded (i.e. (i) & (iv) hold). Then (x_n) converges to $\beta := \sup\{x_n : n \in \mathbb{N}\}$ (which exists in $\mathbb{R}$ by III & (i)).

6. State and prove the result corresponding to 5 for decreasing bounded seq.

7. Show that any convergent seq (with finite limit) is bounded.

Hint for Q5: Let $\varepsilon > 0$. Then $\beta - \varepsilon < x_N$ for some $N \in \mathbb{N}$. Since $(x_n) \uparrow$ it follows that $\forall n \geq N$,
$\beta - \varepsilon < x_N \leq x_n \leq \beta$ so $|x_n - \beta| < \varepsilon$.

Hint for Q7: Take $\varepsilon := 1$. For this ε, $\exists N \in \mathbb{N}$ s.t. $|x_n - l| < \varepsilon = 1 \quad \forall n \geq N$ (since $(x_n) \to l$).
Since $|x_n| - |l| \leq |x_n - l|$ it follows that

$$|x_n| \leq |l| + 1 \quad \forall n \geq N,$$

and hence that

$$|x_n| \leq M \quad \forall n \in \mathbb{N}$$

where $M := \max\{|l|+1, |x_1|, \dots, |x_{N-1}|\}$

Results given in Ex 5, 6 are usually referred ~~to~~ as the Monotone Convergence Theorem.

Th 1. (computation rules & order preserving). Let $c \in \mathbb{R}$

$\lim x_n = x$ & $\lim y_n = y$ (in $\mathbb{R}$).

Then

(i) $\lim (c x_n) = cx$

(ii) $\lim (x_n \pm y_n) = \lim x_n \pm \lim y_n$

(iii) $\lim (x_n y_n) = xy$ ("product rule")

(iv) $\lim (x_n / y_n) = x/y$ provided that $y \neq 0$, $y_n \neq 0 \; \forall n$ (quotient rule for limits)

(v) $\lim |x_n| = |x|$

(vi) $\lim (x_n \vee y_n) = x \vee y$ $\left(\overset{\text{notation}}{=\!=} \max\{x, y\} \right)$

$\lim (x_n \wedge y_n = x \wedge y$ $\left(\overset{\text{notation}}{=\!=} \min\{x, y\} \right)$

(lattice rule for limits)

(vii) If also $x_n \leq z_n \leq y_n \; \forall n$ and $\lim x_n = x = y = \lim y_n$ (say l) then $\lim z_n = l$ (Squeeze Rule).

In particular $|z_n| \leq |x_n| \to 0$ (as $n \to \infty$) $\Rightarrow$ $z_n \to 0$ as $n \to \infty$.

(viii) If also $x_n \leq y_n \; \forall n$ then $x \leq y$ (order-preserving)

(ix) If also $0 \leq x_n$ (so $\sqrt{x_n} \in \mathbb{R}$) $\forall n$ then $0 \leq x$ and $\lim \sqrt{x_n} = \sqrt{x}$

(x) If $x < y$ then $\exists \, N \in \mathbb{N}$ s.t. $x_n < y_n \; \forall \, n \geq N$.

Proof. (i) Can assume that $c \neq 0$ (so $|c| > 0$).
Let $\varepsilon > 0$. Then $\varepsilon/|c|$ is positive and correspondingly $\exists N \in \mathbb{N}$ s.t. $|x_n - x| < \frac{\varepsilon}{|c|}$ $\forall n \geq N$. Hence

$$|cx_n - cx| = |c| \cdot |x_n - x| < |c| \cdot \frac{\varepsilon}{|c|} = \varepsilon \quad \forall n \geq N.$$

(ii) Let $\varepsilon > 0$. Then, for the positive number $\varepsilon_1 = \varepsilon/2$ and $\varepsilon_2 = \varepsilon/2$, $\exists N_1, N_2 \in \mathbb{N}$ s.t.

$$\begin{cases} |x_n - x| < \varepsilon_1 & \forall n \geq N_1 \\ |y_n - y| < \varepsilon_2 & \forall n \geq N_2. \end{cases}$$

Take $N = \max\{N_1, N_2\}$. Then, $\forall n \geq N$, one has $|x_n - x| < \varepsilon_1$ & $|y_n - y| < \varepsilon_2$ and so

$$|(x_n + y_n) - (x + y)| < \varepsilon_1 + \varepsilon_2 = \varepsilon.$$

The proof for $x_n - y_n$ is similar.

(iii) By Q7, $\exists M > 0$ such that $|x|, |x_n|$ $|y|, |y_n|$ (all $n \in \mathbb{N}$) are all bounded by the same M. Let $\varepsilon > 0$. With $\varepsilon_i := \frac{\varepsilon}{2M}$ $(i = 1, 2)$, $\exists N_1, N_2 \in \mathbb{N}$ s.t.

$$|x_n - x| < \varepsilon_1 \quad \forall n \geq N_1,$$
$$|y_n - y| < \varepsilon_2 \quad \forall n \geq N_2.$$

Let $N = \max\{N_1, N_2\}$. Then, $\forall n \geq N$, one has $|x_n - x| < \frac{\varepsilon}{2M}$ & $|y_n - y| < \frac{\varepsilon}{2M}$ so

$$|x_n y_n - xy| = |x_n y_n - x y_n + x y_n - xy|$$
$$\leq |x_n - x| \cdot |y_n| + |x|\,|y_n - y|$$
$$\leq M|x_n - x| + M|y_n - y| < M \cdot \frac{\varepsilon}{2M} + M\frac{\varepsilon}{2M} = \varepsilon$$

Remark. You may also do as follows.

By Q7 (apply to the convergent seq (y_n)) take $M > 0$ s.t. $|y_n| \leq M$ $\forall n$. Let $\varepsilon > 0$.
Then $\exists N_1, N_2 \in \mathbb{N}$ s.t.

$$|x_n - x| < \frac{\varepsilon}{2M} \quad \forall n \geq N_1 \quad \&$$
$$|y_n - y| < \frac{\varepsilon}{2(|x|+1)} \quad \forall n \geq N_2$$

Let $N = N_1 \vee N_2$. Then $\forall n \geq N$ one has

$$|x_n y_n - xy| \leq M|x_n - x| + |x| \cdot \frac{\varepsilon}{2(|x|+1)}$$
$$< \cancel{M} \cdot \frac{\varepsilon}{2\cancel{M}} + \frac{\varepsilon}{2} = \varepsilon$$

(the use of $|x|+1$ rather than $|x|$ is to ensure it is non-zero in the dinominator).

Warning. But the following "argument" is wrong. Let $\varepsilon>0$. Take

$$\varepsilon_1 := \frac{\varepsilon}{2(|y_n|+1)} \ (>0) \ \&$$

$$\varepsilon_2 := \frac{\varepsilon}{2(|x|+1)} \ (>0)$$

Then, $\exists\ N_1, N_2 \in \mathbb{N}$ s.t.

$$|x_n - x| < \varepsilon_1 \quad \forall\ n \geq N_1$$

$$|x_n - y| < \varepsilon_2 \quad \forall\ n \geq N_2$$

(The point is that your ε_1, N_1 depend on n, and hence if you insist to continue you should use ε_1^n, N_1^n say but then when $k \geq N_1^n$, though you have $|x_k - x| < \varepsilon_1 = \frac{\varepsilon}{2(|y_n|+1)}$ you will have difficulty in dealing with

$$|y_k| \cdot |x_k - x|$$

in

$$|x_k y_k - xy| \leq |x_k y_k - x y_k| + |x y_k - xy|$$
$$\leq |y_k| \cdot |x_k - x| + |x|\,|y_k - y|$$

Therefore the <u>common</u> <u>bound</u> for all $|y_n|$ is essential.

(iv). Suffices to show the special case

$$\lim\left(\frac{1}{y_n}\right) = \frac{1}{y} \qquad (*)$$

(then the general case follows by (iii) as

$$x_n/y_n = x_n \cdot \frac{1}{y_n} \quad)$$

For (*), note that

$$\left|\frac{1}{y_n} - \frac{1}{y}\right| = \left|\frac{y-y_n}{y_n y}\right| = \frac{|y_n - y|}{|y_n|\cdot|y|} \qquad (\#)$$

so, this time, we need to find a positive <u>constant</u> c such that $\frac{1}{|y_n|} \leq \frac{1}{c}$, i.e. $|y_n| \geq c$ for all n (or at least $\forall\, n$ starting from a certain term): you want (y_n) to be <u>bounded away</u> from the origin. To do this, one uses the inequality (triangle-inequality!)

$$|a| - |a-b| \leq |b|$$

and, for positive $\frac{|y|}{2}$, $\exists\ N_0 \in \mathbb{N}$ s.t.

$$|y_n - y| < \frac{|y|}{2} \quad \forall\, n \geq N_0$$

and so (applied to y_n, y in place of b, a)

$$|y| - \frac{|y|}{2} < |y| - |y - y_n| \leq |y_n| \quad \forall\, n \geq N_0 .$$

Letting $c := \frac{|y|}{2}$, we have $c \leq |y_n|\ \forall\, n \geq N_0$.

Now, let $\varepsilon > 0$ and consider $\varepsilon' := c|y|\varepsilon$

Since $\lim y_n = y$, $\exists\, N' \in \mathbb{N}$ s.t.

$$|y_n - y| < \varepsilon' = c|y|\varepsilon \quad \forall\, n \geq N'.$$

Let $N := \max\{N_0, N'\}$. Then, $\forall\, n \geq N$, one has (cf (#))

$$\left|\frac{1}{y_n} - \frac{1}{y}\right| \leq \frac{|y_n - y|}{|y_n| \cdot |y|} \leq \frac{|y_n - y|}{c|y|} < \frac{c|y|\varepsilon}{c|y|} = \varepsilon$$

(v). This follows from the definition of limits and the inequality

$$|a| - |b| \leq |a - b| \quad \text{so} \quad \big||a| - |b|\big| \leq |a - b|$$

by symmetry

and hence

$$\big||x_n| - |x|\big| \leq |x_n - x|.$$

(vi). Note that

$$\max\{x_n, y_n\} = \frac{x_n + y_n + |x_n - y_n|}{2} \quad \left(\frac{和+差}{2} = 大数\right)$$

(you can check its validity!). Thus (vi) follows from (i), (ii) & (v).

(vii). Let $\varepsilon > 0$. Since $\lim x_n = l = \lim z_n$, $\exists\, N_1, N_2 \in \mathbb{N}$

s.t.

$|x_n - l| < \varepsilon \quad (\text{so } l-\varepsilon < x_n < l+\varepsilon) \quad \forall\, n \geq N_1$

& $\qquad l-\varepsilon < \quad < l+\varepsilon \quad \forall\, n \geq N_2$.

Let $N = N_1 \vee N_2$. Then $\forall\, n \geq N$ one has

$$l-\varepsilon < x_n \quad \& \quad < l+\varepsilon$$

and it follows from $x_n \leq \quad \leq$ then

$$l-\varepsilon < y_n < l+\varepsilon \qquad (\forall\, n \geq N)$$

$\therefore \lim_n y_n = l$.

(viii) Let $\varepsilon > 0$. Then, as before $\exists\, N \in \mathbb{N}$ s.t.

$$x - \tfrac{\varepsilon}{2} < x_n < x + \tfrac{\varepsilon}{2} \quad \forall\, x \geq N$$

and

$$y - \tfrac{\varepsilon}{2} < y_n < y + \tfrac{\varepsilon}{2} \quad \forall\, x \geq N$$

Since $x_n \leq y_n \ \forall\, n$, it follows that

$$x - \tfrac{\varepsilon}{2} < x_n \leq y_n < y + \tfrac{\varepsilon}{2} \quad \text{with } n = N.$$

and so $\quad x < y + \varepsilon \ (\forall\, \varepsilon > 0)$. Consequently $x \leq y$ (why?)

(ix). Note that

$$\sqrt{x}\,\big|\sqrt{x_n} - \sqrt{x}\big| \leq \big|\sqrt{x_n} - \sqrt{x}\,\big| \cdot \big|\sqrt{x_n} + \sqrt{x}\big| = |x_n - x|.$$

Thus, assuming $\sqrt{x} \neq 0$ (i.e. $x \neq 0$), the result follows from

$$\big|\sqrt{x_n} - \sqrt{x}\,\big| \leq \frac{|x_n - x|}{\sqrt{x}} \longrightarrow 0 \text{ as } n \to \infty.$$

If $x = 0$ then $\forall\, \varepsilon > 0 \ \exists\, N \in \mathbb{N}$ s.t.

$$0 \leq x_n = |x_n - x| < \varepsilon^2 \quad \forall\, n \geq N$$

because $\lim (x_n - x) = 0$ by (ii) and assumptions. Hence

$$\sqrt{x_n} \leq \varepsilon \quad \forall\, n \geq N$$

showing that $\lim \sqrt{x_n} = 0 = x$.

(x) Let $\varepsilon := \frac{y-x}{2} (>0)$. Then $\exists\, N \in \mathbb{N}$ s.t.

$$x_n < x + \frac{y-x}{2} = \frac{x+y}{2} = y - \frac{y-x}{x} < y_n \quad \forall\, n \geq N$$

(you should provide more details!)

Subsequences. Given a sequence

$(x_n) := (x_1, x_2, x_3, \ldots)$

and a sequence

$(y_k) := (y_1, y_2, y_3, \ldots)$

we say that (y_k) is a subsequence of (x_n) if each y_k can be represented in the form

$y_k = x_{n_k}$ (kth term of (y_k) = n_kth term of (x_n))

such that $\mathbb{N} \ni n_k < n_{k+1}$ (so $n_k + 1 \leq n_{k+1}$) $\forall\, k \in \mathbb{N}$:

① $n_1 < n_2 < n_3 < n_4 < \ldots$

By MI, you should be able to show that

② $k \leq n_k \quad \forall\, k \in \mathbb{N}$.

e.g. tails of a sequence (x_n). Fix any $1 \leq N \in \mathbb{N}$, the "N-tail"

$(x_N, x_{N+1}, x_{N+2} \ldots) = (x_{n_k})$

is a subsequence of (x_n), where $n_k := (N-1) + k$.

$(x_n) \to x$ iff its N-tail converges to x

Subsequence Theorem.

Any subsequence of a convergent sequence (x_n) converges (to the same limit say x as that of (x_n)).

Pf. You use ②.

Examples (cf. § 3.2)

1. $\lim\left(\frac{2n+1}{n}\right) = \lim\left(\frac{2+1/n}{n/n}\right) = \frac{2+0}{1} = 2$

2. $\lim\left(\frac{2n}{n^2+1}\right) = \lim\left(\frac{2n/n^2}{\frac{n^2}{n^2}+\frac{1}{n^2}}\right) = \frac{0}{1+0} = 0$

3. $\lim\left(\frac{\sin n}{n}\right) = 0$ because $\left|\frac{\sin n}{n}\right| \leq \frac{1}{n} \to 0$ as $n\to\infty$.

4. $\lim \frac{n}{2^n} = 0$ because $\underset{(1+1)^n}{\overset{\parallel}{2^n}} \geq \frac{n(n-1)}{2!}$ and

$$0 < \frac{n}{2^n} \leq \frac{\cancel{n}}{\cancel{n}(n-1)/2} \to 0 \text{ as } n\to\infty$$

5. Similarly $\lim \frac{n^7}{2^n} = 0$ & $\lim \frac{n^{10000}}{2^n} = 0$.

6. Let $0<r<1$. Then $r^n \to 0$ as $n\to\infty$.

Sol. Let δ be defined $\frac{1}{r} = 1+\delta$ (so $\delta>0$).

Note that $0<r^n = \frac{1}{(1+\delta)^n} \leq \frac{1}{n\delta} \to 0$ as $n\to\infty$.

(Warning: As logarithm not yet introduced, we should not make use of that at present).

7. If (x_n) is a positive seqnce ($x_n>0\ \forall n$) s.t

$\frac{x_{n+1}}{x_n} \leq 0.9\ \forall n$ (so $x_{n+K} \leq (0.9)^K$ & $x_{K+1} \leq (0.9)^K\ \forall K\in\mathbb{N}$

thus, by Example 6 ..., $\lim_n x_n = 0$. (State & prove the result for $c\in(0,1)$ rather than 0.9)

8. If (x_n) is a positive seq such that
$$\lim \frac{x_{n+1}}{x_n} = l < 1.$$
Then $\lim x_n = 0$.
Hint. Take $c \in (l, 1)$. Then by Th 1 (x), $\exists\ N \in \mathbb{N}$ s.t.
$$\frac{x_{n+1}}{x_n} < c \quad \forall\ n \geq N$$
Apply Ex 7 to N-tail $(x_N, x_{N+1}, \cdots)$

Monotone Subsequence Th.
Let (x_n) be a seq of real numbers. Then $\exists$ a monotone subsequence. Consequently we have

Bolzano-Weierstrass Th. Let (x_n) be a bounded sequence of real numbers. Then $\exists$ a convergent subsequence. (by Monotone Subseq. & Monotone Conv. Theorem).

proof of Monotone Subseq. Let call call the k^{th} term x_k of (x_n) a peak term (with peak-index k) if
$$x_k \geq x_j \quad \forall\ j \geq k$$
(that is x_k dominates all subsequence terms after it in the seq (x_n)).

Two cases to be considered : Either there are finitely many peak terms or " " infinitely many peak terms :

Case 1. Suppose there are only finitely many peaks so $\exists$ $N \in \mathbb{N}$ so that x_N is the last peak-term, i.e x_n is not a peak-term $\forall\, n > N$. Pick $n_1 = N+1$. Since $N+1$ is not peaked, $\exists\, n_2 > n_1$ s.t

$$x_{n_2} > x_{n_1}$$

Since n_2 is not peaked, $\exists\, n_3 > n_2$ s.t

$$x_{n_3} > x_{n_2} \quad (> x_{n_1})$$

Inductively one has (x_{n_k}) with

$$n_{k+1} > n_k \quad \forall\, k$$

&

$$x_{n_{k+1}} > x_{n_k} \quad \forall\, k$$

Thus (x_{n_k}) is a (strictly) increasing subseq of (x_n).

Case 2. There are infinitely many peaks

$$n_1 < n_2 < n_3 < \cdots$$

(each of them is a peak) so with

$$x_{n_1} \geq x_{n_2} \geq x_{n_3} \geq \cdots$$

(by definition of peaks). Thus (x_{n_k}) is a decreasing subsequence of (x_n).

Th (Cauchy Criterion). A seq (x_n) converges (~~to~~ a limit in $\mathbb{R}$) iff it is Cauchy.

Proof. Since "$\Rightarrow$" part already done we only prove "$\Leftarrow$" part. So suppose (x_n) is Cauchy. Then it is bounded (pl. supply a proof) and hence it follows from the Bolzano-Weierstrass Theorem that it has a convergent subsequence, say $\lim_k x_{n_k} = x \in \mathbb{R}$. Let $\varepsilon > 0$. Then $\exists\, K \in \mathbb{N}$ s.t

$$|x_{n_k} - x| < \frac{\varepsilon}{2} \quad \forall\, k \geq K \qquad (1)$$

and also $\exists\, N \in \mathbb{N}$ s.t

$$|x_n - x_m| < \frac{\varepsilon}{2} \quad \forall\, m, n \geq N. \qquad (2)$$

Take $k \in \mathbb{N}$ s.t $k \geq K, N$ (so $n_k \geq N$)

Then

$$|x_{n_k} - x| < \varepsilon/2 \quad \& \quad |x_n - x_{n_k}| < \varepsilon/2 \ \forall n \geq N$$

and it follows from the triangle inequality that

$$|x_n - x| \leq |x_n - x_{n_k}| + |x_{n_k} - x| < \varepsilon/2 + \varepsilon/2 = \varepsilon \quad \forall \ n \geq N$$

that is $\lim x_n = x$